

Sigma model from $SU(1, 1|2)$ spin chain

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Abstract

We derive the coherent state representation of the integrable spin chain Hamiltonian with supersymmetry group $SU(1, 1|2)$. By the use of a projected Hamiltonian onto bosonic states, we give explicitly the action of the Hamiltonian on $SU(2) \times SL(2)$ coherent states. Passing to the continuous limit, we find that the corresponding bosonic sigma model is the sum of the known $SU(2)$ and $SL(2)$ ones, and thus it gives a string spinning fast on $S^1_{\phi_1} \times S^1_{\varphi_1} \times S^1_{\varphi_2}$ in $AdS_5 \times S^5$. The full sigma model on the supercoset $SU(1, 1|2)/SU(1|1)^2$ is given.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 2 | The coherent state | 4 |
| 3 | Hamiltonian action on $SU(1, 1 2)$ states | 6 |
| 4 | $SU(1, 1 2)$ sigma model | 8 |
| 4.1 | $SU(2) \times SL(2)$ truncated Hamiltonian | 8 |
| 4.2 | Continuous limit | 9 |
| 5 | Bosonic string action | 11 |
| 6 | Conclusion | 12 |
| A | $\mathfrak{su}(1, 1 2)$ algebra | 13 |

1 Introduction

The AdS/CFT correspondence [1, 2, 3] between strings on anti-de Sitter (AdS) spaces and boundary gauge theories has generated much interest in recent years. One of the most studied examples relates string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) gauge theory. String states in the bulk are dual to gauge invariant operators in the boundary and an increasing holographic dictionary between their correlation functions has been derived. In [4], this holographic correspondence was established in the neighborhood of null geodesics of $AdS_5 \times S^5$, where the geometry looks like a pp-wave [5]. On such a geometry, string theory is known to be solvable [6, 7], while on the gauge theory side it corresponds to SYM operators with large \mathcal{R} -symmetry charge J . In [8]-[17], the authors studied the fluctuations around semiclassical spinning strings and showed that, there also, energies of classical string solutions matched anomalous dimensions of SYM operators with large charges. Recently the matrix model approach to the anomalous dimension matrix in $\mathcal{N} = 4$ SYM theory was considered [18].

On the gauge theory side, the planar one-loop anomalous dimensions in $\mathcal{N} = 4$ SYM turned out to be described by integrable spin chain Hamiltonians [19, 20, 21]. Thanks to integrability, strong progress was then achieved in the comparison of the spectrums on both sides, as it allowed to use powerful Bethe Ansatz techniques¹. We refer the reader to [30]-[35] for extensive reviews and citations and to [36, 37] for recent important results on the subject where the authors give the Bethe Ansätze for the full $SU(2, 2|4)$ group in the thermodynamic limit at one and higher loops.

In the continuous (BMN) limit, where $J \gg 1$, the spin chains can be identified with the worldsheet of closed strings². The spin chain excitations give then the string profile in the symmetry group taken as target space and the spin chain Hamiltonian describes the

¹When non-planar corrections are included [22]-[29], there arises an interesting question, about the possibility of an extended or modified notion of (quasi) integrability.

²For a treatment of the problem of fermion doubling and the BMN correspondence, see [38, 39, 40].

dynamic of the string. As for the BMN case, the perturbative regime of SYM is accessible to this limit and accordingly, string and spin chain sigma model actions should agree. By the use of a coherent vector description of the spin chain, this was shown to be the case in [41] for the $SU(2)$ subsector of the theory. Extension to the whole $SO(6)$ and its other compact subgroups was then performed in [42]-[46]. The non compact $SL(2)$ case was studied in [44, 47, 48]. This study was extended to the supersymmetric sectors $SU(1|2)$ [49], $SU(1,1|1)$ [50] and $SU(2|3)$ [49, 51]. In this last paper, the authors also discussed on a generalisation to the full $SU(2,2|4)$.

In all cases, semiclassical spinning string states were identified with coherent states. These are built from spin chain states by acting with the coset $G/H_{|0\rangle}$ on a vacuum $|0\rangle$, with G the subsector studied and $H_{|0\rangle}$ the stabilizer of G by respect to $|0\rangle$. Because of the properties of the coherent states, one can go to a path integral formulation without loosing any information of the initial theory. Passing then to the continuous limit along the spin chain gives the sigma model. We refer the reader to [52]-[65] for further developments in this subject.

We will focus in this paper on the $SU(1,1|2)$ sector of the theory. This sector is interesting as it generalizes the two simpler bosonic sectors $SU(2)$ and $SL(2)$. Each of these two subsectors carry information from the two main bosonic parts of $SU(2,2|4)$ which are $SO(2,4)$ and $SO(6)$, and interact between themselves via supersymmetric charges. The $SU(1,1|2)$ sector corresponds to SYM operators made out of two scalars carrying different $SU(2)$ charges and two fermions, plus derivatives along a fixed direction ($SL(2)$ charge). The corresponding Bethe Ansatz in the thermodynamic limit is discussed in details in [36, 37]. The sector is non compact, and its representations are thus infinite dimensional. We first derive a coherent state representation of the spin chain Hamiltonian. Like in the simpler $SU(1,1|1)$ case [50], the Hamiltonian results to have a non-linear form, with a logarithmic term. But moreover, it cannot be expressed, as it was the case for the $SU(1,1|2)$ subgroups, in terms of the square on the superspace of a single vector $(\vec{n}_2 - \vec{n}_1)^2$ built from coherent states. This makes its fermionic part quite involved. However, by passing to the continuous limit where expressions simplify a lot, the fermions reassemble in the simple square $\partial\vec{n}^2$, like in [49, 50, 51]. The so obtained sigma model should then correspond to a string moving on the supercoset $SU(1,1|2)/SU(1|1)^2$. We check that this is at least the case for the bosonic action that turn out to be the sum of the $SU(2)$ and $SL(2)$ sigma models [41, 44, 47].

The paper is organized as follows. In Section 2 we build the $SU(1,1|2)$ coherent state. In Section 3, we give an expression of the two-site Hamiltonian suitable for acting on our coherent states. In Section 4 we derive the sigma model on the group manifold $SU(1,1|2)/SU(1|1)^2$. As the fermionic part of the Hamiltonian action on coherent states resulted into a too long and too complicated expression to be presented in a paper (69 terms), we do this in two steps. First, we focus in Subsection 4.1 on a truncated Hamiltonian acting on $SU(2) \times SL(2)$ spin chains. It consists in the projection of the full $SU(1,1|2)$ Hamiltonian onto pure bosonic states. Its action on two coherent states is given explicitly and gives rise to a strange mix between the $SU(2)$ and $SL(2)$ separate actions. Then in Subsection 4.2, we go to the continuous limit, putting back the fermions. In Section 5 we show that the bosonic sigma model arises when considering a string spinning fast on S_ϕ^1 on AdS_5 and $S_{\varphi_1}^1 \times S_{\varphi_2}^1$ in S^5 . Finally in Section 6 we summarize our results. Appendix A collect the definitions in terms of oscillators of states and charges,

as well as the commutation relations of the $SU(1, 1|2)$ algebra.

2 The coherent state

Coherent states are defined by the choice of a group G and a vacuum $|0\rangle$ in a representation \mathcal{R} of the group. We denote by $H_{|0\rangle}$ the corresponding stabilizer subgroup, *i.e.* the group of elements of G that leave $|0\rangle$ invariant up to a phase. The coherent state is then defined by the action of a finite group element of $g \in G/H_{|0\rangle}$ on $|0\rangle$. With G chosen as being $SU(1, 1|2)$, we will take $|0\rangle$ to be the physical vacuum $|\phi_0\rangle$. The generators for the algebra \mathfrak{g} are taken to be

$$T_A = (J_0, R_0, P, K, R_{23}, R_{32}, Q_2, Q_3, S_2, S_3, \dot{Q}_2, \dot{Q}_3, \dot{S}_2, \dot{S}_3) .$$

Conventions and details about the algebra and its singleton representation are given in appendix A. The stabilizer subgroup $H_{|\phi_0\rangle}$ is generated by

$$H_{|\phi_0\rangle} = \{J_0, R_0, Q_3, S_3, \dot{Q}_2, \dot{S}_2\} = SU(1|1)^2 \quad .$$

We chose the coherent state to be

$$|\vec{n}\rangle = g(\vec{n}) |\phi_0\rangle = e^{zR_{3,2} - \bar{z}R_{2,3}} e^{uP - \bar{u}K} e^{-\xi Q_2 - \bar{\xi} S_2} e^{-\theta \dot{Q}_3 - \bar{\theta} \dot{S}_3} |\phi_0\rangle \quad , \quad (2.1)$$

where $z = \psi e^{i\varphi}$ and $u = \rho e^{i\phi}$. It is parameterized by four real parameters $\rho, \psi, \phi, \varphi$ and two complex Grassmann variables ξ and θ . The coherent state $|\mathbf{n}\rangle$ of the full spin chain writes as the direct product

$$|\mathbf{n}\rangle = |\vec{n}_1\rangle \otimes |\vec{n}_2\rangle \dots |\vec{n}_J\rangle \quad (2.2)$$

where each $|\vec{n}_k\rangle$ denotes a coherent state (2.1) with its own parameters $\rho_k(t), \psi_k(t), \dots$, and describes the spin chain excitation at site k and time t .

The spin chain action will then be given in terms of the spin chain Hamiltonian H by

$$S = - \int dt \left(i \langle \mathbf{n} | \partial_t | \mathbf{n} \rangle + \hat{\lambda} \langle \mathbf{n} | H | \mathbf{n} \rangle \right) , \quad (2.3)$$

which, after taking the continuous limit $J \rightarrow \infty$, will lead us to the sigma model.

The first task consists in expanding the coherent states in the basis $\{|\phi_m\rangle, |Z_m\rangle, |\lambda_m\rangle, |\mu_m\rangle\}$, with ϕ, Z being scalar fields, λ, μ being fermions, and $m = 0, 1, 2, \dots$ labelling the number of derivatives among a fixed direction. This is done using Table 4 and gives the following expression for the coherent state expression :

$$\begin{aligned} |\vec{n}\rangle = \sum_{m=0}^{\infty} \frac{e^{i m \phi} \tanh^m \rho}{\cosh \rho} & \left[\left(1 + \frac{1}{2} \xi \bar{\xi} + \frac{1}{2} \theta \bar{\theta} + \frac{1}{4} \xi \bar{\xi} \theta \bar{\theta} \right) (\cos \psi |\phi_m\rangle + e^{i\varphi} \sin \psi |Z_m\rangle) \right. \\ & + \frac{e^{-i\phi} \xi \theta}{\cosh \rho} \left(\frac{m}{\sinh \rho} - \sinh \rho \right) (\cos \psi |Z_m\rangle - e^{-i\varphi} \sin \psi |\phi_m\rangle) \\ & \left. - \frac{\xi}{\cosh \rho} \left(1 + \frac{1}{2} \theta \bar{\theta} \right) |\mu_m\rangle - \frac{\theta}{\cosh \rho} \left(1 + \frac{1}{2} \xi \bar{\xi} \right) |\lambda_m\rangle \right] . \quad (2.4) \end{aligned}$$

The coherent states $|\vec{n}\rangle$, although not orthogonal, are normalized :

$$\langle \vec{n} | \vec{n} \rangle = 1 .$$

They are over-complete, *i.e.* they fulfill a resolution of unity :

$$\mathbb{I} = \frac{2j_{sl2} - 1}{4\pi^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin 2\psi d\psi \int_0^\infty \sinh 2\rho d\rho \int d\xi d\bar{\xi} d\theta d\bar{\theta} |\vec{n}\rangle \langle \vec{n}| . \quad (2.5)$$

Here, j_{sl2} is the spin by respect to the $sl(2)$ subalgebra. In our choice of states, it equals $\frac{1}{2}$ for the bosons and 1 for the fermions. In an arbitrary j_{sl2} representation, the coherent state infinite expansion depends explicitly on j_{sl2} [66], and the integral over $d\rho$ gives a $(2j_{sl2} - 1)^{-1}$ factor. Therefore, when acting on bosons, the integral should be seen as acting on coherent states $|\vec{n}(j_{sl2})\rangle$ in the limit $j_{sl2} \rightarrow \frac{1}{2}$ (see [47] for details).

It is possible to associate to each coherent state a point $\vec{n} = \{n_{J_0}, n_{R_0}, \dots\}$ in the superspace by defining

$$n_A \equiv \langle \vec{n} | T_A | \vec{n} \rangle . \quad (2.6)$$

The action of the charges is given in Table 4 and leads, after summation in (2.4) to

$$\vec{n} : \begin{cases} n_{J_0} = \frac{1}{2} \cosh 2\rho (1 - \xi \bar{\xi} - \theta \bar{\theta}) \\ n_{R_0} = \frac{1}{2} \cos 2\psi (1 + \xi \bar{\xi} + \theta \bar{\theta}) \\ n_P = \overline{n_K} = e^{-i\phi} \cosh \rho \sinh \rho (1 - \xi \bar{\xi} - \theta \bar{\theta}) \\ n_{R_{23}} = \overline{n_{R_{32}}} = e^{i\varphi} \cos \psi \sin \psi (1 + \xi \bar{\xi} + \theta \bar{\theta}) \\ n_{Q_2} = \overline{n_{S_2}} = \cos \psi \cosh \rho \bar{\xi} - e^{-i(\phi+\varphi)} \sin \psi \sinh \rho \theta \\ n_{Q_3} = \overline{n_{S_3}} = e^{i\varphi} \sin \psi \cosh \rho \bar{\xi} + e^{-i\phi} \cos \psi \sinh \rho \theta \\ n_{\dot{Q}_2} = \overline{n_{\dot{S}_2}} = e^{-i\phi} \cos \psi \sinh \rho \xi - e^{i\varphi} \sin \psi \cosh \rho \bar{\theta} \\ n_{\dot{Q}_3} = \overline{n_{\dot{S}_3}} = e^{-i(\phi+\varphi)} \sin \psi \sinh \rho \xi + \cos \psi \cosh \rho \bar{\theta} \end{cases} . \quad (2.7)$$

The resulting vector is null $n^A n_A = 0$ with respect to the metric g_{AB} given by

$$\begin{aligned} n_A m^A &= g^{AB} n_A m_B \\ &= \frac{1}{2} n_{J_0} m_{J_0} - \frac{1}{2} n_{R_0} m_{R_0} - \frac{1}{2} n_P m_K - \frac{1}{2} n_{R_{23}} m_{R_{32}} - \frac{1}{2} n_{Q_i} m_{S_i} - \frac{1}{2} n_{\bar{Q}_i} m_{\bar{S}_i} + \text{h.c.} \end{aligned} \quad (2.8)$$

Contrary to the usual case, the metric is not defined through the Killing metric, which here vanishes identically, but is given by the Casimir of the group (see Appendix A).

The first (Wess-Zumino) term in (2.3) can be easily evaluated by taking the derivative of (2.4) and then performing the infinite sum. It has the simple form

$$i \langle \mathbf{n} | \partial_t | \mathbf{n} \rangle = \sum_{\text{sites } k} \left[-\sinh^2 \rho \partial_t \phi - \sin^2 \psi \partial_t \varphi + \frac{i}{2} (\bar{\xi} D_t \xi + \xi \bar{D}_t \bar{\xi} + \bar{\theta} D_t \theta + \theta \bar{D}_t \bar{\theta}) \right]_k \quad (2.9)$$

with the covariant derivative defined as

$$D_a \equiv \partial_a + i C_a , \quad C_a \equiv \sinh^2 \rho \partial_a \phi - \sin^2 \psi \partial_a \varphi . \quad (2.10)$$

Evaluating the second term in (2.3) requires much more work. In order to compute the average of the Hamiltonian between two spin chain coherent states $|\mathbf{n}\rangle$, one should first express the Hamiltonian action on the basis $\{\phi_m, Z_m, \lambda_m, \mu_m\}$. This is done in the next section.

3 Hamiltonian action on $SU(1, 1|2)$ states

We here rewrite the $SU(1, 1|2)$ two-sites harmonic Hamiltonian given by Beisert in [20] in the $\{\phi_m, Z_m, \lambda_m, \mu_m\}$ basis. The total Hamiltonian on the spin chain is then given by the summation over all two neighboring sites along the spin chain :

$$H = \sum_{k=1}^J H_{k,k+1} \quad .$$

Omitting the site's k indices, a two-sites state is given by $|A_m, B_n\rangle \equiv |A_m\rangle \otimes |B_n\rangle$, where A_m, B_m stand for any of the $\{\phi_m, Z_m, \lambda_m, \mu_m\}$. We have also to introduce some other definitions, accounting for the usual harmonic number $h(m)$, a permutation operator³ \mathcal{P} , raising/lowering operators $\mathcal{T}_{1,2}^\pm$, and a supersymmetric operator \mathcal{Q} :

$$\begin{aligned} h(m) &= \sum_{i=1}^m \frac{1}{i} \quad , \quad \mathcal{P} |A_m, B_n\rangle = |B_m, A_n\rangle \quad , \\ \mathcal{T}_1^\pm |A_m, B_n\rangle &= |A_{m\pm 1}, B_n\rangle \quad , \quad \mathcal{T}_2^\pm |A_m, B_n\rangle = |A_m, B_{n\pm 1}\rangle \quad , \\ \mathcal{Q} |A_m, B_n\rangle &= |\mathcal{Q}(A)_m, \mathcal{Q}(B)_n\rangle \quad , \quad \mathcal{Q}(\phi) = \lambda \quad , \quad \mathcal{Q}(\lambda) = \phi, \quad \mathcal{Q}(Z) = -\mu, \quad \mathcal{Q}(\mu) = -Z \quad . \end{aligned}$$

By looking to the general shapes arising from the action of the harmonic Hamiltonian [20] on states with a small number of derivatives, it is possible to get a whole picture of its general action in the $SU(1, 1|2)$ subsector. One ends with three possible cases :

- Boson/boson interaction

$$\begin{aligned} H_{12} |A_k, B_l\rangle &= \left[h(k) + h(l) + \frac{1 - \mathcal{P}}{k + l + 1} \right] |A_k, B_l\rangle \\ &+ \sum_{i=1}^k \left[\frac{1 - \mathcal{P}}{k + l + 1} \left(1 - \frac{\mathcal{Q} \mathcal{T}_2^-}{l + i} \right) - \frac{1}{i} \right] |A_{k-i}, B_{l+i}\rangle \\ &+ \sum_{j=1}^l \left[\frac{1 - \mathcal{P}}{k + l + 1} \left(1 + \frac{\mathcal{Q} \mathcal{T}_1^-}{k + j} \right) - \frac{1}{j} \right] |A_{k+j}, B_{l-j}\rangle \end{aligned} \quad (3.1)$$

³Let us point out here that \mathcal{P} permutes only letters, not their $sl(2)$ charges, and it does *not* take into account supersymmetric gradings, as it is common in the literarcy.

where letters A and B stand either for ϕ or Z .

- Fermion/fermion interaction

$$\begin{aligned}
H_{12} |A_k, B_l\rangle &= \left[h(k+1) + h(l+1) - \frac{1-\mathcal{P}}{k+l+2} (1 + (k+1) \mathcal{Q} \mathcal{T}_1^+ - (l+1) \mathcal{Q} \mathcal{T}_2^+) \right] |A_k, B_l\rangle \\
&+ \sum_{i=1}^k \left[\frac{(l+1)(1-\mathcal{P})}{k+l+2} \left(\mathcal{Q} \mathcal{T}_2^+ - \frac{1}{l+i+1} \right) + \frac{1}{l+i+1} - \frac{1}{i} \right] |A_{k-i}, B_{l+i}\rangle \quad (3.2) \\
&+ \sum_{j=1}^l \left[\frac{(k+1)(1-\mathcal{P})}{k+l+2} \left(-\mathcal{Q} \mathcal{T}_1^+ - \frac{1}{k+j+1} \right) + \frac{1}{k+j+1} - \frac{1}{j} \right] |A_{k+j}, B_{l-j}\rangle
\end{aligned}$$

where letters A and B stand either for λ or μ .

- Boson/fermion interaction

$$\begin{aligned}
H_{12} |A_k, B_l\rangle &= \left[h(k + F_A) + h(l + F_B) - \left(\frac{F_B}{1+k} + \frac{F_A}{1+l} \right) \mathcal{P} \right] |A_k, B_l\rangle \\
&+ \sum_{i=1}^k \left(\frac{F_B - F_A \mathcal{P}}{l+i+1} - \frac{1}{i} \right) |A_{k-i}, B_{l+i}\rangle \quad (3.3) \\
&+ \sum_{j=1}^l \left(\frac{F_A - F_B \mathcal{P}}{k+j+1} - \frac{1}{j} \right) |A_{k+j}, B_{l-j}\rangle
\end{aligned}$$

where letters A and B stand for any letter, F_A and F_B being their respective supersymmetric grading ($F_\phi = F_Z \equiv 0$ and $F_\lambda = F_\mu \equiv 1$). The condition $|F_A - F_B| = 1$ is assumed.

One can remark here that in going from the oscillator picture to precise states $\{\phi_m, Z_m, \lambda_m, \mu_m\}$, the harmonic Hamiltonian [20] loses its very concise and elegant form. However, it gains two nice advantages : first, the conditions in the number of oscillators become implicit ; second, although more complicated, its *computation* can be done much faster. Indeed, the two-site harmonic Hamiltonian, because of its sum on possible permutations on oscillator sites, is computable in exponential time. Its derivation in the form given here is computable in linear time by respect to the number of oscillator composing the initial states. Therefore, while computations of states as *e.g.* $H_{12} |\phi_{10}, \mu_{10}\rangle$ were reaching the capacities of normal computers, they become here immediate. Another expression computable in quadratic time was given in [67] as the anti-commutator of lowering/increasing length operators.

4 $SU(1,1|2)$ sigma model

The next task consists in taking the average of the Hamiltonian by two-sites coherent states :

$$\langle \vec{n}_1(\rho_1, \psi_1, \dots), \vec{n}_2(\rho_2, \psi_2, \dots) | H_{12} | \vec{n}_1(\rho_1, \psi_1, \dots), \vec{n}_2(\rho_2, \psi_2, \dots) \rangle . \quad (4.1)$$

The computation is extremely long and tedious, as one has to act with equations (3.1), (3.2), (3.3) on two coherent states (2.4), and then perform a double or triple sum. It is however still doable with the extensive use of a computer.

In the $SU(1,1|1)$ case, it was possible to express the Grassmann variables appearing in (4.1) in a very compact form as they just summed up with the bosonic ones into a logarithm. We could not get a simple expression in the considered $SU(1,1|2)$ case. It seems that this is a direct consequence of the supersymmetric mixing between $su(2)$ and $sl(2)$ subalgebras. For the sake of simplicity, we will therefore just give the bosonic part of the two-site Hamiltonian average. We will return to fermionic considerations when taking the continuous limit, where expressions simplifies a lot.

4.1 $SU(2) \times SL(2)$ truncated Hamiltonian

In the $SU(2)$ subsector, the average (4.1) was linear in $(\vec{n}_1 - \vec{n}_2)^2$ [41], while this same square appeared⁴ in a logarithm in the $SL(2)$ [44, 47] and $SU(1,1|1)$ [50] subsectors. As we will see in this subsection, the average Hamiltonian in the $SU(1,1|2)$ sector cannot be expressed as a function of $(\vec{n}_1 - \vec{n}_2)^2$ only.

As all fermions appear in the coherent state (2.4) together with a Grassmann variable, it is clear that the bosonic part of the average Hamiltonian (4.1) will be given by (i) restricting ourselves to coherent states with Grassmann variables set to zero and (ii) taking as Hamiltonian the projection of the bosonic/bosonic interaction (3.1) onto bosonic states. Computation of (ii) is straightforward, and leads to

$$\begin{aligned} H_{12}^{\text{bosonic}} |A_k, B_l\rangle &= [h(k) + h(l)] |A_k, B_l\rangle - \sum_{\substack{i=0 \\ i \neq k}}^{k+l} \frac{1}{|k-i|} |A_i, B_{k+l-i}\rangle \\ &\quad + \frac{1 - \mathcal{P}}{k+l+1} \sum_{i=0}^{k+l} |A_i, B_{k+l-i}\rangle \\ &= H_{12}^{sl(2)} |A_k, B_l\rangle + \frac{H_{12}^{su(2)}}{k+l+1} \sum_{i=0}^{k+l} |A_i, B_{k+l-i}\rangle . \end{aligned} \quad (4.2)$$

Here, A and B stand for ϕ or Z . $H_{12}^{sl(2)}$ is the $sl(2)$ two-sites Hamiltonian found in [20], while $H_{12}^{su(2)} \equiv 1 - \mathcal{P}$ is the usual Heisenberg $XXX_{1/2}$ two-sites Hamiltonian. The Hamiltonian (4.2) appears as an Hamiltonian on $SU(2) \times SL(2)$ spin chains. By construction it commutes with all the bosonic charges, and the spin j states $|j\rangle$ defined in appendix A are still eigenvectors : $H_{12}^{\text{bosonic}} |j\rangle = 2h(j) |j\rangle$ [20].

⁴It is always assumed that square acting on vectors use the corresponding group metrics g_{AB} .

Computing the average of H_{12}^{bosonic} in the bosonic sector of $SU(1,1|2)$ then leads to

$$\langle \vec{n}_1, \vec{n}_2 | H_{12}^{\text{bosonic}} | \vec{n}_1, \vec{n}_2 \rangle = \left(1 - \frac{(\vec{n}_2 - \vec{n}_1)_{su2}^2}{(\vec{n}_2 - \vec{n}_1)_{sl2}^2} \right) \log(1 - (\vec{n}_2 - \vec{n}_1)_{sl2}^2) \quad , \quad (4.3)$$

where $(\vec{n}_2 - \vec{n}_1)_{sl2}^2$ and $(\vec{n}_2 - \vec{n}_1)_{su2}^2$ are exactly the terms appearing in the $SU(2)$ and $SL(2)$ subsectors respectively ! To be more precise, one has

$$(\vec{n}_2 - \vec{n}_1)_{sl2}^2 = (\vec{n}_2 - \vec{n}_1)^2 \Big|_{\psi, \varphi, \theta, \xi \rightarrow 0} \quad \text{and} \quad (\vec{n}_2 - \vec{n}_1)_{su2}^2 = -(\vec{n}_2 - \vec{n}_1)^2 \Big|_{\rho, \phi, \theta, \xi \rightarrow 0}$$

so that

$$\begin{aligned} (\vec{n}_2 - \vec{n}_1)_{sl2}^2 &= \frac{1}{2} (1 - \cosh 2\rho_1 \cosh 2\rho_2 + \cos(\phi_1 - \phi_2) \sinh 2\rho_1 \sinh 2\rho_2) \quad , \\ (\vec{n}_2 - \vec{n}_1)_{su2}^2 &= \frac{1}{2} (1 - \cos 2\psi_1 \cos 2\psi_2 - \cos(\varphi_1 - \varphi_2) \sin 2\psi_1 \sin 2\psi_2) \quad . \end{aligned} \quad (4.4)$$

The limit to the subsectors $SU(2)$ and $SL(2)$ appears clearly, as they impose respectively $(\vec{n}_2 - \vec{n}_1)_{sl2}^2 \rightarrow 0$ and $(\vec{n}_2 - \vec{n}_1)_{su2}^2 \rightarrow 0$.

Computing the square of the full coherent vectors with g_{AB} , one finds

$$(\vec{n}_2 - \vec{n}_1)^2 = (\vec{n}_2 - \vec{n}_1)_{sl2}^2 - (\vec{n}_2 - \vec{n}_1)_{su2}^2 + \text{fermions} \quad .$$

Therefore, just looking to (4.3), one concludes that it will not be possible to express the average of the full $SU(1,1|2)$ Hamiltonian just in terms of $(\vec{n}_2 - \vec{n}_1)^2$, as it was the case in the $SU(2)$ and $SU(1,1|1)$ subsectors : the square on “coherent vectors” is cut into two pieces, in order to give a mix of precedent $SU(2)$ and $SL(2)$ found shapes.

4.2 Continuous limit

The fermionic part of (4.1) is much more involved. It contains 69 different terms without any clear structure. Even the quadratic terms appear with coefficients that mix trigonometric and hyperbolic functions in an highly non trivial way. We found no simplification neither rewriting (4.3) with the Ansatz

$$\langle \vec{n}_1, \vec{n}_2 | H_{12} | \vec{n}_1, \vec{n}_2 \rangle = \left(1 - \frac{(\vec{n}_2 - \vec{n}_1)_{su2}^2 + \text{fermions}_1}{(\vec{n}_2 - \vec{n}_1)_{sl2}^2 + \text{fermions}_2} \right) \log(1 - (\vec{n}_2 - \vec{n}_1)_{sl2}^2 - \text{fermions}_1) \quad ,$$

that was suggested by the bosonic average.

Fortunately, everything simplifies a lot in the limit $\vec{n}_2 \rightarrow \vec{n}_1$. The result is, with fermionic part included,

$$\langle \vec{n}_1, \vec{n}_2 | H_{12} | \vec{n}_1, \vec{n}_2 \rangle = -\epsilon^2 g^{AB} \delta n_A \delta n_B + \mathcal{O}(\epsilon^3)$$

where $\vec{n}_2 = \vec{n}_1 + \epsilon \vec{\delta n}$. Like all results previously found for $SU(1,1|2)$ subsectors, it

appears that in the continuous limit, the Hamiltonian is just the square, made with the corresponding superspace metric, of the first derivative along the spin chain.

Summing up over the spin chain sites $k = 1, \dots, J$ and passing to the continuous limit, one finally gets the Hamiltonian of the sigma model :

$$\begin{aligned}
\langle \mathbf{n} | H | \mathbf{n} \rangle &= -\frac{1}{J} \int d\sigma \, g^{AB} \partial_\sigma n_A \partial_\sigma n_B \\
&= \frac{1}{J} \int d\sigma \left(\bar{D}_\sigma \bar{\xi} D_\sigma \xi + \bar{D}_\sigma \bar{\theta} D_\sigma \theta + \mathbf{e}^2 (1 + 2 \bar{\xi} \xi \bar{\theta} \theta) - (\bar{\mathbf{e}}_{\mathbf{A}} \mathbf{e}_{\mathbf{A}} - \bar{\mathbf{e}}_{\mathbf{B}} \mathbf{e}_{\mathbf{B}})(\bar{\xi} \xi + \bar{\theta} \theta) \right. \\
&\quad \left. + 2 \bar{\mathbf{e}}_{\mathbf{A}} \bar{\mathbf{e}}_{\mathbf{B}} \theta \xi + 2 \mathbf{e}_{\mathbf{A}} \mathbf{e}_{\mathbf{B}} \bar{\xi} \bar{\theta} \right).
\end{aligned} \tag{4.5}$$

In this last expression, the covariant derivative is given by (2.10) and

$$\begin{aligned}
\mathbf{e}_{\mathbf{A}} &= e^{i\phi} \left(\partial_\sigma \rho + \frac{i}{2} \sinh 2\rho \, \partial_\sigma \phi \right) \quad , \quad \mathbf{e}_{\mathbf{B}} = e^{i\varphi} \left(\partial_\sigma \psi + \frac{i}{2} \sin 2\psi \, \partial_\sigma \varphi \right) \quad , \\
\mathbf{e} &= \mathbf{e}_{\mathbf{A}} \bar{\mathbf{e}}_{\mathbf{A}} + \mathbf{e}_{\mathbf{B}} \bar{\mathbf{e}}_{\mathbf{B}} = (\partial_\sigma \rho)^2 + \frac{1}{4} \sinh^2 2\rho \, (\partial_\sigma \phi)^2 + (\partial_\sigma \psi)^2 + \frac{1}{4} \sin^2 2\psi \, (\partial_\sigma \varphi)^2 \quad .
\end{aligned} \tag{4.6}$$

It is now possible to get the full sigma model action by plugging (2.9) and (4.5) into (2.3). Its decomposition into bosonic and fermionic parts writes :

$$S_B = -J \int d\sigma dt \left(-\sinh^2 \rho \, \partial_t \phi - \sin^2 \psi \, \partial_t \varphi + \frac{\hat{\lambda}}{J^2} \mathbf{e}^2 \right) \quad , \tag{4.7}$$

$$\begin{aligned}
S_F &= -J \int d\sigma dt \left[\frac{i}{2} (\bar{\xi} D_t \xi + \xi \bar{D}_t \bar{\xi} + \bar{\theta} D_t \theta + \theta \bar{D}_t \bar{\theta}) + \frac{\hat{\lambda}}{J^2} \left(\bar{D}_\sigma \bar{\xi} D_\sigma \xi + \bar{D}_\sigma \bar{\theta} D_\sigma \theta \right. \right. \\
&\quad \left. \left. + (\bar{\mathbf{e}}_{\mathbf{B}} \mathbf{e}_{\mathbf{B}} - \bar{\mathbf{e}}_{\mathbf{A}} \mathbf{e}_{\mathbf{A}})(\bar{\xi} \xi + \bar{\theta} \theta) + 2 \bar{\mathbf{e}}_{\mathbf{A}} \bar{\mathbf{e}}_{\mathbf{B}} \theta \xi + 2 \mathbf{e}_{\mathbf{A}} \mathbf{e}_{\mathbf{B}} \bar{\xi} \bar{\theta} + 2 \mathbf{e}^2 \bar{\xi} \xi \bar{\theta} \theta \right) \right] \quad .
\end{aligned} \tag{4.8}$$

The bosonic action S_B is exactly the sum of the $SU(2)$ and $SL(2)$ actions obtained in [41] and [44, 47]. This structure may appear as a direct consequence from the fact that the bosonic part of $su(1, 1|2)$ is the direct product $sl(2) \times su(2)$. However, one should remark that the Hamiltonian projection on bosonic states (4.2) is not the sum of the Hamiltonians restricted to these two subsectors. As it is proved in section 5, S_B corresponds to the bosonic action of a string spinning fast in $S_{\phi_1} \times S_{\varphi_1} \times S_{\varphi_2}$, with S_{ϕ_1} in AdS_5 and $S_{\varphi_1} \times S_{\varphi_2}$ in S^5 .

The action S_F appears to be more complicated, as it is through the fermions that $SU(2)$ and $SL(2)$ sectors interact. As one could have expected, it is not quadratic in fermions, although the quartic term in Grassmann variables is just proportional to the bosonic Hamiltonian \mathbf{e}^2 . Because of the mixed coefficients in front of the Grassmann variables, it seems that getting the corresponding superstring description will be but a hard task. For example, in the $SU(2|3)$ sector were one deals with three complex scalars and two complex fermions, the Grassmann variables appear in the sigma model with the same factor \mathbf{e}^2 [49].

5 Bosonic string action

As one can expect from the bosonic action (4.7), the “fast spinning” limit to the dual bosonic string is a mix of the $SU(2)$ and $SL(2)$ found limits [41, 44, 47]. What happens here is that $SU(2)$ and $SL(2)$ parts will share a fast spinning circle in S^5 .

The bosonic part of Polyakov action describing a string moving on $AdS_5 \times S^5$ can be written as

$$S = \frac{R^2}{4\pi\alpha'} \int g_{MN} (\partial_\tau X^M \partial_\tau X^N - \partial_\sigma X^M \partial_\sigma X^N) \quad (5.1)$$

with

$$ds^2 = g_{MN} dX^M dX^N = ds_{AdS_5}^2 + ds_{S^5}^2$$

and

$$\begin{aligned} ds_{AdS_5}^2 &= d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2) , \\ ds_{S^5}^2 &= d\gamma^2 + \cos^2 \gamma d\varphi_3^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_1^2 + \sin^2 \psi d\varphi_2^2) . \end{aligned} \quad (5.2)$$

Imposing $\gamma = \frac{\pi}{2}$, $\theta = 0$ and making the change of variables

$$\phi_1 = \phi + t , \quad \varphi_1 = \hat{\varphi} + t , \quad \varphi_2 = \varphi + \hat{\varphi} + t , \quad (5.3)$$

the metric (5.2) rewrites as

$$\begin{aligned} ds^2 &= 2 dt (d\hat{\varphi} + \sinh^2 \rho d\phi + \sin^2 \psi d\varphi) + d\hat{\varphi}^2 + 2 \sin^2 \psi d\varphi d\hat{\varphi} \\ &\quad + d\rho^2 + \sinh^2 \rho d\phi^2 + d\psi^2 + \sin^2 \psi d\varphi^2 . \end{aligned} \quad (5.4)$$

As usual, we make the light-cone gauge choice $t = \kappa \tau$ and take the limit $\kappa \rightarrow +\infty$, keeping $\kappa \partial_\tau X^M$ fixed for the other coordinates.

The action (5.1) should also satisfy the Virasoro constraints. To leading order in κ , the first of them reads

$$g_{MN} \partial_\tau X^M \partial_\sigma X^N = \kappa (\sin^2 \psi \partial_\sigma \varphi + \sinh^2 \rho \partial_\sigma \phi + \partial_\sigma \hat{\varphi}) = 0 , \quad (5.5)$$

and can be used to solve for $\partial_\sigma \hat{\varphi}$.

Evaluating the action (5.1) with the metric (5.4) and using (5.5), one gets to leading order in κ

$$\begin{aligned} S &= -\frac{R^2 \kappa}{2\pi\alpha'} \int d\sigma dt \left(-\sinh^2 \rho \partial_t \phi - \sin^2 \psi \partial_t \varphi - \partial_t \hat{\varphi} \right. \\ &\quad \left. + \frac{1}{2\kappa^2} [(\partial_\sigma \rho)^2 + \frac{1}{4} \sinh^2 \rho (\partial_\sigma \phi)^2 + (\partial_\sigma \psi)^2 + \frac{1}{4} \sin^2 \psi (\partial_\sigma \varphi)^2] \right) . \end{aligned}$$

Identifying

$$J = \frac{R^2 \kappa}{2\pi\alpha'} \quad \text{and} \quad \tilde{\lambda} = \frac{R^4}{8\pi^2 \alpha'^2} , \quad (5.6)$$

the string action gives back the bosonic spin chain sigma model (4.7), up to the full time derivative $\partial_t \hat{\varphi}$.

In the original variables, the limit $\kappa \rightarrow +\infty$ corresponds to $\partial_\tau \phi_1 \approx \partial_\tau \varphi_1 \approx \partial_\tau \varphi_2 \approx k$, so the string spins fast on $S_{\varphi_1}^1 \times S_{\varphi_2}^1 \in S^5$ and $S_{\phi_1}^1 \in AdS_5$. The most simple non trivial solutions for the classical equations of motion is then given by a multi-spinning string folded or circular by respect to the ψ coordinate [41] and folded by respect to the ρ coordinate [47].

6 Conclusion

The sigma model arising from $SU(1,1|2)$ spin chains was derived. Doing so, a truncated Hamiltonian on $SU(2) \times SL(2)$ spin chains appeared. This truncated Hamiltonian leads to a one to one correspondence between long bosonic spin chains states in the $SU(1,1|2)$ planar subsector of $\mathcal{N} = 4$ SYM gauge theory at one loop, and bosonic strings spinning fast on two circles in S^5 and one circle in AdS_5 . The resulting action is the sum of the sigma models arising from $SU(2)$ spin chains and $SL(2)$ ones. The average of this Hamiltonian between two neighboring bosonic coherent states is not anymore expressible in terms of the full “coherent vector” square $(\vec{n}_{k+1} - \vec{n}_k)^2$ but rather as a mixing between pure $SU(2)$ and $SL(2)$ terms :

$$\langle \mathbf{n} | H^{\text{bosonic}} | \mathbf{n} \rangle = \sum_{k=1}^L \left(1 - \frac{(\vec{n}_{k+1} - \vec{n}_k)_{su2}^2}{(\vec{n}_{k+1} - \vec{n}_k)_{sl2}^2} \right) \log(1 - (\vec{n}_{k+1} - \vec{n}_k)_{sl2}^2) \quad .$$

When one takes into account the fermionic part, the correspondence to super-strings seems much more involved. Such difficulties should increase in enlarging to bigger sectors of the full theory, and a way out could be to build the sigma models not in terms of precise coordinates, but rather in terms of more general expressions with constraints given by the coset structure as proposed in [51]. Another possibility could be to reason in terms of Cartan forms L^a , as the string action on $AdS_5 \times S^5$ in terms of these is known [68]. Indeed, as soon as the Hamiltonian in the continuous limit is proportional to $(\partial_\sigma \vec{n})^2$, as it is the case here, it is possible to express the spin chain sigma model in a G -invariant form as [50]

$$S = -J \int d^2\sigma \left[i L_t^A n_A - \frac{\hat{\lambda}}{J^2} (L_\sigma^B f_{BA}{}^C n_C)^2 \right],$$

where $f_{BA}{}^C$ are the structure constants of the considered group G .

The two-loop Hamiltonian of $SU(1,1|2)$ sector was given recently in [67] in terms of rising/lowering length operators. At higher loops, the Hamiltonian starts to change the length of the spin chain. However, such interactions may be absent in the continuous limit, as argued in [69]. Then it would be possible to compute the sigma model up to two loops and see how it would match, at least for the bosonic part, with the fast spinning string.

Another issue is to ask oneself the integrability of the truncated, bosonic, Hamiltonian (4.2). Although we checked that the simple Ansatz $Q = \sum_{k=1}^L [H_{k+2,k+1}^{\text{bosonic}}, H_{k+1,k}^{\text{bosonic}}]$ for the next higher charge does not commute with H^{bosonic} , integrability may be retained in some involved way. It would then provide us an example of an integrable $SU(2) \times SL(2)$ spin chain.

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Appendix

In this appendix we collect the commutation relations and details on the “singleton” representations of the superalgebra $\mathfrak{g} = \mathfrak{su}(1, 1|2)$. A singleton corresponds to a subsector of the $\mathcal{N} = 4$ SYM multiplet that closes under \mathfrak{g} . Here we adopt the oscillator description (see [20] for details). In this formalism, elementary SYM fields (the singleton of $\mathfrak{psu}(2, 2|4)$) are represented by acting on a Fock vacuum $|0\rangle$ with bosonic $(a_\alpha, b_{\dot{\alpha}})$ and fermionic oscillators c_A , $(\alpha, \dot{\alpha} = 1, 2, A = 1, \dots, 4)$. Physical states satisfy the condition

$$C = n_a - n_b + n_c = 2 \quad (\text{A.0})$$

with n_a, n_b, n_c denoting the number of oscillators of a given type.

The closed subalgebras of $\mathfrak{su}(2, 2|4)$ are defined by restricting the range of $\alpha, \dot{\alpha}, A$.

A $\mathfrak{su}(1, 1|2)$ algebra

The algebra $su(1, 1|2)$ is built in terms of bilinears of two bosonic (a, b) and two fermionic (c_2, c_3) oscillators. It consists of an $su(2)$ charge R_0 , an $sl(2)$ charge J_0 , Lorentz translation and boost P and K , $su(2)$ R-symmetry rotation operators R_{23} and R_{32} , and eight fermionic supertranslations and superboosts Q_i, \dot{Q}_i, S_i and \dot{S}_i . Here, $i = 2, 3$, in order to follow the notations of [20]. We choose as physical vacuum the state $|\phi_0\rangle = c_1^\dagger c_2^\dagger |0\rangle$.

States in the singleton representation are given by

$$\begin{aligned} |\phi_m\rangle &= \frac{1}{m!} (a^\dagger b^\dagger)^m |\phi_0\rangle && \Leftrightarrow \frac{1}{m!} \mathcal{D}^m \phi_0 \\ |Z_m\rangle &= \frac{1}{m!} (a^\dagger b^\dagger)^m c_3^\dagger c_2 |\phi_0\rangle && \Leftrightarrow \frac{1}{m!} \mathcal{D}^m Z_0 \\ |\lambda_m\rangle &= \frac{1}{m!} (a^\dagger b^\dagger)^m b^\dagger c_3^\dagger |\phi_0\rangle && \Leftrightarrow \frac{1}{m!} \mathcal{D}^m \lambda_0 \\ |\mu_m\rangle &= \frac{1}{m!} (a^\dagger b^\dagger)^m a^\dagger c_2 |\phi_0\rangle && \Leftrightarrow \frac{1}{m!} \mathcal{D}^m \mu_0 \end{aligned} \quad (\text{A.1})$$

and correspond to two scalar fields ϕ_0, Z_0 and two fermions λ_0, μ_0 , together with their m -derivatives along a fixed direction. In order to get rid of square roots in expressions, the states are normalized according to :

$$\langle \phi_m | \phi_m \rangle = \langle Z_m | Z_m \rangle = 1, \quad \langle \lambda_m | \lambda_m \rangle = \langle \mu_m | \mu_m \rangle = m + 1 \quad .$$

The algebra in this case is non-compact and the representations are infinite-dimensional.

The generators in terms of oscillators are given by

$$\begin{aligned} R_{23} &= c_2^\dagger c_3 & Q_i &= a^\dagger c_i \\ R_{32} &= c_3^\dagger c_2 & \dot{Q}_i &= b^\dagger c_i^\dagger & J_0 &= \frac{1}{2} (1 + a^\dagger a + b^\dagger b) \\ P &= a^\dagger b^\dagger & S_i &= a c_i^\dagger & R_0 &= \frac{1}{2} (c_2^\dagger c_2 - c_3^\dagger c_3) \\ K &= a b & \dot{S}_i &= b c_i \end{aligned}$$

| $\{\downarrow, \rightarrow\}$ | S_2 | Q_3 | S_3 | \dot{Q}_2 | \dot{S}_2 | \dot{Q}_3 | \dot{S}_3 |
|-------------------------------|-------------|----------|-------------|-------------|-------------|-------------|-------------|
| Q_2 | $J_0 + R_0$ | | R_{32} | P | | | |
| S_2 | | R_{23} | | | K | | |
| Q_3 | | | $J_0 - R_0$ | | | P | |
| S_3 | | | | | | | K |
| \dot{Q}_2 | | | | $J_0 - R_0$ | | | $-R_{23}$ |
| \dot{S}_2 | | | | | $-R_{32}$ | | |
| \dot{Q}_3 | | | | | | $J_0 + R_0$ | |

Table 1: Fermionic anticommutators. Here and below, redundant subdiagonal terms are omitted.

| $[\downarrow, \rightarrow]$ | Q_2 | S_2 | Q_3 | S_3 | \dot{Q}_2 | \dot{S}_2 | \dot{Q}_3 | \dot{S}_3 |
|-----------------------------|-------------------|-------------------|------------------|-------------------|------------------------|-------------------------|-------------------------|-------------------------|
| P | | $-\dot{Q}_2$ | | $-\dot{Q}_3$ | | $-Q_2$ | | $-Q_3$ |
| K | \dot{S}_2 | | \dot{S}_3 | | S_2 | | S_3 | |
| R_{23} | $-Q_3$ | | | S_2 | | $-\dot{S}_3$ | \dot{Q}_2 | |
| R_{32} | | S_3 | $-Q_2$ | | \dot{Q}_3 | | | $-\dot{S}_2$ |
| J_0 | $\frac{1}{2}Q_2$ | $-\frac{1}{2}S_2$ | $\frac{1}{2}Q_3$ | $-\frac{1}{2}S_3$ | $\frac{1}{2}\dot{Q}_2$ | $-\frac{1}{2}\dot{S}_2$ | $\frac{1}{2}\dot{Q}_3$ | $-\frac{1}{2}\dot{S}_3$ |
| R_0 | $-\frac{1}{2}Q_2$ | $\frac{1}{2}S_2$ | $\frac{1}{2}Q_3$ | $-\frac{1}{2}S_3$ | $\frac{1}{2}\dot{Q}_2$ | $-\frac{1}{2}\dot{S}_2$ | $-\frac{1}{2}\dot{Q}_3$ | $\frac{1}{2}\dot{S}_3$ |

Table 2: Fermionic/Bosonic commutators

| $[\downarrow, \rightarrow]$ | K | R_{23} | R_{32} | J_0 | R_0 |
|-----------------------------|---------|----------|----------|-------|-----------|
| P | $-2J_0$ | | | $-P$ | |
| K | | | | K | |
| R_{23} | | | $2R_0$ | | $-R_{23}$ |
| R_{32} | | | | | R_{32} |

Table 3: Bosonic commutators

| $\downarrow \rightarrow \rangle$ | $ \phi_m\rangle$ | $ Z_m\rangle$ | $ \lambda_m\rangle$ | $ \mu_m\rangle$ |
|------------------------------------|---------------------------------|------------------------------|------------------------------|---------------------------|
| Q_2 | $ \mu_m\rangle$ | 0 | $-(m+1) Z_{m+1}\rangle$ | 0 |
| S_2 | 0 | $-\lambda_{m-1}\rangle$ | 0 | $(m+1) \phi_m\rangle$ |
| Q_3 | 0 | $ \mu_m\rangle$ | $(m+1) \phi_{m+1}\rangle$ | 0 |
| S_3 | $ \lambda_{m-1}\rangle$ | 0 | 0 | $(m+1) Z_m\rangle$ |
| \dot{Q}_2 | 0 | $-\lambda_m\rangle$ | 0 | $(m+1) \phi_{m+1}\rangle$ |
| \dot{S}_2 | $ \mu_{m-1}\rangle$ | 0 | $-(m+1) Z_m\rangle$ | 0 |
| \dot{Q}_3 | $ \lambda_m\rangle$ | 0 | 0 | $(m+1) Z_{m+1}\rangle$ |
| \dot{S}_3 | 0 | $ \mu_{m-1}\rangle$ | $(m+1) \phi_m\rangle$ | 0 |
| P | $(m+1) \phi_{m+1}\rangle$ | $(m+1) Z_{m+1}\rangle$ | $(m+1) \lambda_{m+1}\rangle$ | $(m+1) \mu_{m+1}\rangle$ |
| K | $m \phi_{m-1}\rangle$ | $m Z_{m-1}\rangle$ | $(m+1) \lambda_{m-1}\rangle$ | $(m+1) \mu_{m-1}\rangle$ |
| R_{23} | 0 | $ \phi_m\rangle$ | 0 | 0 |
| R_{32} | $ Z_m\rangle$ | 0 | 0 | 0 |
| J_0 | $(m+\frac{1}{2}) \phi_m\rangle$ | $(m+\frac{1}{2}) Z_m\rangle$ | $(m+1) \lambda_m\rangle$ | $(m+1) \mu_m\rangle$ |
| R_0 | $\frac{1}{2} \phi_m\rangle$ | $-\frac{1}{2} Z_m\rangle$ | 0 | 0 |

Table 4: Action of $SU(1,1|2)$ charges

The $sl(2)$ and $su(2)$ charges J_0 and R_0 give the Cartan of the group. Non vanishing commutation relations are given in the Tables 1, 2, 3. The action of the charges upon states is given in Table 4.

A single trace SYM operator of length J is given by the tensor products of J singletons : we take J copies of the oscillators a, b, c and impose the condition (A.0) at each site. The symmetry algebra is then taken to be the diagonal $SU(1, 1|2)$ algebra

$$\tilde{T}_A = \sum_{k=1}^J (T_A)_k \quad (\text{A.2})$$

with $(T_A)_k$ acting on the k^{th} site.

The Killing metric of $SU(1, 1|2)$ vanishes identically. However, it is still possible to define a metric g_{AB} through the Casimir of the algebra, which is given here by :

$$\hat{C}_2 = g^{AB} T_A T_B = J_0^2 - R_0^2 - \frac{1}{2}\{P, K\} - \frac{1}{2}\{R_{23}, R_{32}\} - \frac{1}{2}[Q_i, S_i] - \frac{1}{2}[\dot{Q}_i, \dot{S}_i] . \quad (\text{A.3})$$

\hat{C}_2 defines the spin j by

$$\hat{C}_2 = j(j+1) \mathbb{I} .$$

In the singleton representation, the spin vanishes : $j = 0$ for all the 1-site states (A.1). Spin j representations arise then in the tensor product of two singletons and the corresponding highest weight states can be written as follows :

$$|j\rangle_{1,2} = \sum_{i=0}^j (-1)^i \binom{j}{i} |\phi_{j-i}\rangle_1 \otimes |\phi_i\rangle_2 . \quad (\text{A.4})$$

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